

KHOVANOV-ROZANSKY HOMOLOGY AND THE BRAID INDEX OF A KNOT

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ABSTRACT. We construct knots for whom the new Khovanov-Rozansky-Morton-Franks-Williams inequality gives a sharp bound for its braid index; however, the classical Morton-Franks-Williams inequality fails to do so. We also construct infinitely many knots for which the KR-MFW inequality fails to detect the braid indices.

1. INTRODUCTION

The Alexander's theorem states that any knot or link is isotopic to the closure of a braid. We can measure the complexity of knot \mathcal{K} by the minimal possible number of braid strands, which is called the *braid index* $b_{\mathcal{K}}$. Morton [11], Franks and Williams [5] found an inequality which gives a lower bound for the braid index.

Let us first fix some notation.

Let $\mathcal{K} \subset S^3$ be an oriented knot or link and $\mathcal{B}_{\mathcal{K}}$ be the infinite set of closed braid diagrams of \mathcal{K} . For an oriented closed braid diagram D , let b_D denote the number of the braid strands and w_D the *writhe*, which is the number of positive crossings minus the number of negative crossings of D . Then the *braid index* of \mathcal{K} is $b_{\mathcal{K}} := \min\{b_D | D \in \mathcal{B}_{\mathcal{K}}\}$.

We adopt the following definition and normalization of the HOMFLYPT polynomial $P_{\mathcal{K}}(a, q)$ defined by the skein relation;

$$(1.1) \quad aP_{\mathcal{K}_-}(a, q) - a^{-1}P_{\mathcal{K}_+}(a, q) = (q - q^{-1})P_{\mathcal{K}_0}(a, q) \quad \text{and} \quad P_{\text{unknot}}(a, q) = 1.$$

Let $d_{\pm}(\mathcal{K})$ be the maximal (resp. minimal) a -degree of $P_{\mathcal{K}}(a, q)$.

Now we state the Morton-Franks-Williams (MFW) inequality:

Theorem 1.1. [5] [11] [**Morton-Franks-Williams inequality**] *For any closed braid diagram $D \in \mathcal{B}_{\mathcal{K}}$ of knot or link \mathcal{K} we have*

$$w_D - b_D + 1 \leq d_{-}(\mathcal{K}) \leq d_{+}(\mathcal{K}) \leq w_D + b_D - 1.$$

Moreover

$$\frac{d_{+}(\mathcal{K}) - d_{-}(\mathcal{K})}{2} + 1 \leq b_{\mathcal{K}}.$$

Khovanov-Rozansky homology [9] is a *categorification* of HOMFLYPT polynomial. In this paper, we use the *reduced* HOMFLY homology $\overline{H}^{i,j,k}(\mathcal{K})$ of \mathcal{K} introduced by Rasmussen [13]. The graded Euler characteristic of the reduced HOMFLY homology is equal to the normalized HOMFLYPT polynomial (1.1);

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \overline{H}^{i,j,k}(\mathcal{K}) = P_{\mathcal{K}}(a, q).$$

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Dunfield, Gukov, Rasmussen [3] and Wu [16] found Khovanov-Rozansky homology version of MFW inequality. We call it *KR-MFW inequality*.

Theorem 1.2. [**KR-MFW-inequality**] [3], [16]. *Let \mathcal{K} be a knot or a link and*

$$\delta_+(\mathcal{K}) := \max\{j \mid \overline{H}^{i,j,k}(\mathcal{K}) \neq 0, \text{ for some } i, k\},$$

$$\delta_-(\mathcal{K}) := \min\{j \mid \overline{H}^{i,j,k}(\mathcal{K}) \neq 0, \text{ for some } i, k\}.$$

Then, for any closed braid diagram $D \in \mathcal{B}_{\mathcal{K}}$ of \mathcal{K} we have

$$w_D - b_D + 1 \leq \delta_-(\mathcal{K}) \leq \delta_+(\mathcal{K}) \leq w_D + b_D - 1 \quad \text{i.e.,} \quad \frac{\delta_+(\mathcal{K}) - \delta_-(\mathcal{K})}{2} + 1 \leq b_{\mathcal{K}}.$$

Definition 1.3. The MFW (resp. KR-MFW) inequality is called *sharp* on \mathcal{K} if there exists $D \in \mathcal{B}_{\mathcal{K}}$ such that equalities $w_D - b_D + 1 = d_-(\mathcal{K})$ (resp. $\delta_-(\mathcal{K})$) and $d_+(\mathcal{K}) = w_D + b_D - 1$ (resp. $\delta_+(\mathcal{K})$) hold.

Since $\delta_-(\mathcal{K}) \leq d_-(\mathcal{K}) \leq d_+(\mathcal{K}) \leq \delta_+(\mathcal{K})$, we have the following.

Proposition 1.4. The sharpness of the MFW (resp. KR-MFW) inequality implies

$$\frac{d_+(\mathcal{K}) - d_-(\mathcal{K})}{2} + 1 = b_{\mathcal{K}} \quad (\text{resp.} \quad \frac{\delta_+(\mathcal{K}) - \delta_-(\mathcal{K})}{2} + 1 = b_{\mathcal{K}}).$$

Elrifai [4] has enumerated all the 3-braids on which the MFW inequality is non-sharp.

Theorem 1.5. [4] [**Elrifai's example**] *On all knots and links of braid index = 3 the MFW inequality is sharp except*

$$\begin{aligned} \mathcal{K}_k &:= \text{the braid closure of } (\sigma_1 \sigma_2 \sigma_2 \sigma_1)^{2k} \sigma_1 \sigma_2^{-2k-1} \\ \mathcal{L}_k &:= \text{the braid closure of } (\sigma_1 \sigma_2 \sigma_2 \sigma_1)^{2k+1} \sigma_1 \sigma_2^{-2k+1} \end{aligned}$$

for $k \in \mathbb{N}$ and their mirror images $\overline{\mathcal{K}}_k, \overline{\mathcal{L}}_k$.

As the Euler characteristic of the KR homology gives us the HOMFLYPT polynomial, KR-homology contains more information than HOMFLYPT polynomial. It is interesting to find concrete examples that show the “gap” between KR-homology and HOMFLYPT polynomial. Elrifai's example seem to be natural candidates to see such gap. In fact, we have:

Theorem 1.6. *Let*

$$\mathcal{K}^* := \mathcal{K}_1 = \text{the braid closure of } (\sigma_1 \sigma_2 \sigma_2 \sigma_1)^2 \sigma_1 \sigma_2^{-3}.$$

On \mathcal{K}^ and its mirror image $\overline{\mathcal{K}^*}$ the MFW-inequality is not sharp but the KR-MFW inequality is sharp.*

These are the first examples which show that Khovanov-Rozansky homology is “stronger” than HOMFLYPT polynomial in terms of detecting the braid index.

However, we will also see that Khovanov-Rozansky homology is not almighty. We study an obstruction of sharpness of the KR-MFW inequality and give infinitely many (and also first known) examples in Theorem 1.7 whose braid index KR-homology fails to detect.

Let $BM_{x,y,z,w}$ where $x, y, z, w \in \mathbb{Z}$ be the closure of the 4-strand braid

$$\sigma_1^x \sigma_2^y \sigma_3^{-1} \sigma_2^z \sigma_1^w \sigma_2 \sigma_3 \sigma_2 \sigma_2 \sigma_3.$$

It has been known [7] that the set of BM-braids contains the five knots 9_{42} , 9_{49} , 10_{132} , 10_{150} , 10_{156} , on which the MFW inequality is not sharp [6]. Furthermore, the diagram contains infinitely many four tuples (x, y, z, w) where the MFW inequality is not sharp [7]. A parallel result holds for the KR-MFW inequality:

Theorem 1.7. *There are infinitely many four tuples (x, y, z, w) such that the KR-MFW inequality is not sharp on $BM_{x,y,z,w}$.*

Elrifai's examples have another interesting feature regarding to the generalized Jones' conjecture ([10], [7]) and the maximal Bennequin number Conjecture as we state below. See [6] p.357 for Jones' original conjecture.

Conjecture 1.8. [generalized Jones' conjecture] Let $\Phi : \mathcal{B}_K \rightarrow \mathbb{N} \times \mathbb{Z}$ be a map with $\Phi(D) := (b_D, w_D)$ for $D \in \mathcal{B}_K$. Then there exists a unique $w_K \in \mathbb{Z}$ such that

$$(1.2) \quad \Phi(\mathcal{B}_K) = \{(b_K + x + y, w_K + x - y) \mid x, y \in \mathbb{N}\},$$

which is the subset of the infinite shaded region shown in Figure 1.

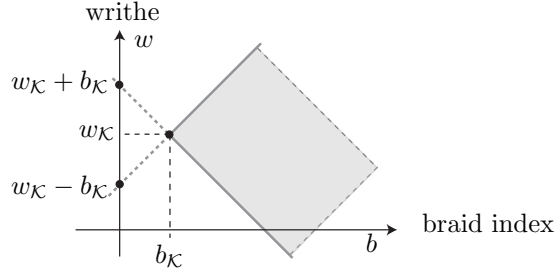


FIGURE 1. The region of braid representatives of \mathcal{K}

We can apply Conjecture 1.8 to contact geometry. Bennequin [1] proved that any transversal knot in the standard contact structure (S^3, ξ_{std}) can be identified with a closed braid in \mathbb{R}^3 . The *Bennequin (self linking) number* $\beta_D := w_D - b_D$ of braid diagram $D \in \mathcal{B}_K$ is an invariant of transversal knots and links in (S^3, ξ_{std}) . Then we denote the *maximal Bennequin number* of \mathcal{K} by $\beta_K := \{\beta_D \mid D \in \mathcal{B}_K\}$. Conjecture 1.8 implies the following:

Conjecture 1.9. [The maximal Bennequin number conjecture] The maximal *Bennequin number* β_K of \mathcal{K} is realized at the minimal braid index and $\beta_K = w_K - b_K$.

By Definition 1.3, it follows that:

Proposition 1.10. The sharpness of the (KR-)MFW inequality implies Conjectures 1.8 and 1.9.

Namely, the two inequalities $w_D - b_D + 1 \leq d_-$ and $d_+ \leq w_D + b_D - 1$ in Theorem 1.1 correspond to the two boundary lines of the infinite region in Figure 1.

Conjecture 1.8 holds for many classes of knots and links: Franks and Williams [5] have proved the sharpness of the MFW inequality of knot and link with a braid representative of full positive twists with a positive word $(\Delta^{2n}P)$ including

unlinks, torus links and Lorenz links. Murasugi [12] has affirmed the sharpness for alternating fibred knots and links and 2-bridge knots and links. Jones [6] verified the sharpness for all the knots of less than or equal to 10 crossing in the standard knot table except $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$. Thus, Conjecture 1.8 holds for these knots and links by Proposition 1.10.

In [8], we have proved Conjecture 1.8 for $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$, by computing deficits of the MFW inequality of cabled knots.

In [7], more general results are given: If Conjecture 1.8 holds for \mathcal{K}, \mathcal{L} then Conjecture 1.8 also holds for the connect sum $\mathcal{K} \# \mathcal{L}$ and the (p, q) -cable of \mathcal{K} .

Thanks to Elrifai (Theorem 1.5) we can improve the study of Conjectures 1.8 and 1.9 for the set of 3-braids B_3 . Let $B'_3 := B_3 \setminus \{ \mathcal{K}_k, \mathcal{L}_k, \overline{\mathcal{K}_k}, \overline{\mathcal{L}_k} \mid k \in \mathbb{N} \}$. Rudolph's slice Bennequin inequality [14] will play an important role to prove the following:

Theorem 1.11. *Conjecture 1.8 holds for B'_3 , \mathcal{K}_k and $\overline{\mathcal{K}_k}$ for $k \in \mathbb{N}$.*

Theorem 1.12. *Conjecture 1.9 holds for B'_3 , $\mathcal{K}_k, \mathcal{L}_k$ and $\overline{\mathcal{K}_k}$ for $k \in \mathbb{N}$.*

The rest of the paper is organized as follows. In Section 2, we compute Khovanov-Rozansky homology of \mathcal{K}^* and prove Theorem 1.6. In Section 3, we discuss about non-sharpness of the KR-MFW inequality and prove Theorem 1.7. In Section 4, we prove Theorems 1.11 and 1.12.

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2. PROOF OF THEOREM 1.6

We first recall some of the works of Rasmussen. Let $\sigma(\mathcal{K})$ be the *signature* of \mathcal{K} . A knot \mathcal{K} is called *KR-thin* if $\overline{H}^{i,j,k}(\mathcal{K}) = 0$ whenever $i + j + k \neq \sigma(\mathcal{K})$. If \mathcal{L} be a 2-component link, we call it *KR-thin* if $\overline{H}_N(\mathcal{L})$, the *totally reduced* homology of \mathcal{L} , is thin for all sufficiently large $N > 1$. Denote $\delta := i + j + k$. In [13], Rasmussen proved the following.

Theorem 2.1. (1) *Let $\mathcal{K}_+, \mathcal{K}_-, \mathcal{K}_0$ be links or knots differ by a single site. If $\mathcal{K}_-, \mathcal{K}_0$ are KR-thin and*

$$\det \mathcal{K}_- + 2 \det \mathcal{K}_0 = \det \mathcal{K}_+$$

then \mathcal{K}_+ is also KR-thin (Corollary 7.7 of [13]).

- (2) *The connect sum of two KR-thin knots is also KR-thin (Corollary 7.9 of [13]).*
- (3) *Among the knots with less than or equal to 9 crossings, only $8_{19}, 9_{42}, 9_{43}, 9_{47}$ are not KR-thin (Proposition 7.10 of [13]).*

Now we prove Theorem 1.6.

Proof of Theorem 1.6. We specify a resolution of \mathcal{K}^* . For simplicity let $n := \sigma_n$ ($n = 1, 2$) the generator of the Artin's braid group B_3 and $\bar{n} := \sigma_n^{-1}$. Let

$$\begin{aligned} \mathcal{K}^* = \mathcal{K}_+ &= \{1, 2, 2, 1, 1, 2, 2, 1, 1, \bar{2}, \bar{2}, \bar{2}\} \\ \mathcal{K}_- &= \{1, 2, 2, 1, 1, 2, 2, 1, \bar{1}, \bar{2}, \bar{2}, \bar{2}\} = \{1, 2, 2, 1, 1, \bar{2}\} = \overline{5_2} \text{ mirror image} \\ \mathcal{K}_0 &= \{1, 2, 2, 1, 1, 2, 2, 1, \bar{2}, \bar{2}, \bar{2}\} \\ \mathcal{K}_{0-} &= \{1, 2, 2, 1, 1, 2, \bar{2}, 1, \bar{2}, \bar{2}, \bar{2}\} = \{1, 2, 2, 1, 1, 1, \bar{2}, \bar{2}, \bar{2}\} \\ \mathcal{K}_{00} &= \{1, 2, 2, 1, 1, 2, 1, \bar{2}, \bar{2}, \bar{2}\} = \overline{5_2} \\ \mathcal{K}_{0--} &= \{1, 2, \bar{2}, 1, 1, 1, \bar{2}, \bar{2}, \bar{2}\} = \{1, 1, 1, 1, \bar{2}, \bar{2}, \bar{2}\} = T_{2,-3} \# T_{2,4} \\ \mathcal{K}_{0-0} &= \{1, 2, 1, 1, 1, \bar{2}, \bar{2}, \bar{2}\} = \text{unknot} \end{aligned}$$

where $T_{p,q}$ is the (p, q) -torus knot or link.

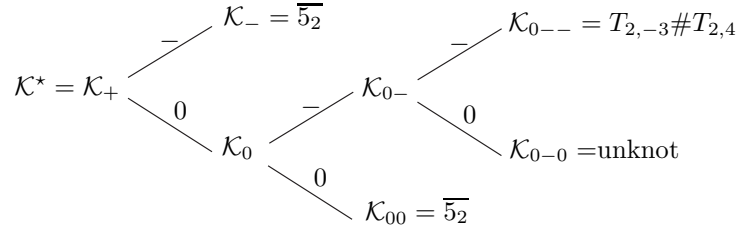


FIGURE 2. Resolution of \mathcal{K}^*

Thanks to Theorem 2.1.2 and 2.1.3, knots $\overline{5_2}$ and $T_{2,-3} \# T_{2,4}$ are KR-thin. Since

$$\det(\mathcal{K}_{0--}) + 2 \det(\mathcal{K}_{0-0}) = 12 + 2 = 14 = \det(\mathcal{K}_{0-}),$$

Theorem 2.1.1 tells that \mathcal{K}_{0-} is also KR-thin and have the following table.

		σ	\det	KR-thin	
\mathcal{K}_+		knot	2	7	non-thin
$\mathcal{K}_- = \mathcal{K}_{00}$	$\overline{5_2}$	knot	2	7	thin
\mathcal{K}_0		link	1	0	non-thin
\mathcal{K}_{0-}		link	1	14	thin
\mathcal{K}_{0--}	$T_{2,-3} \# T_{2,4}$	link	1	12	thin
\mathcal{K}_{0-0}	unknot	knot	0	1	thin

Let $\overline{H}_N(\mathcal{K})$ be the reduced $sl(N)$ homology group defined by Khovanov and Rozansky [9]. It satisfies

$$\sum_{I,J} (-1)^J q^I \dim \overline{H}_N^{I,J}(\mathcal{K}) = P_{\mathcal{K}}(q^N, q).$$

Let M be a free module of rank = 4 whose graded Poincare polynomial is $(q/t + t/q)^2$. Using Proposition 7.6 of [13] we have the following skein exact sequences.

$$(2.1) \quad \longrightarrow \overline{H}_N(\mathcal{K}_{0-}) \longrightarrow \overline{H}_N(\mathcal{K}_{00} = \overline{5_2}) \otimes M \longrightarrow \overline{H}_N(\mathcal{K}_0) \longrightarrow \overline{H}_N(\mathcal{K}_{0-}) \longrightarrow$$

$$(2.2) \quad \longrightarrow \overline{H}_N(\mathcal{K}_- = \overline{5_2}) \longrightarrow \overline{H}_N(\mathcal{K}_0) \longrightarrow \overline{H}_N(\mathcal{K}_+) \longrightarrow \overline{H}_N(\mathcal{K}_- = \overline{5_2}) \longrightarrow$$

Their HOMFLYPT polynomials satisfy:

$$\begin{aligned}
(2.3) \quad P_{\mathcal{K}_+} &= a^8(-q^4 - 1 - q^{-4}) + a^6(q^6 + q^2 + q^{-2} + q^{-6}) \\
P_{\overline{5_2}} &= -a^6 + a^4(q^2 - 1 + q^{-2}) + a^2(q^2 - 1 + q^{-2}) \\
(q - q^{-1})^2 P_{\overline{5_2}} &= a^6(-q^2 + 2 - q^{-2}) + a^4(q^4 - 3q^2 + 4 - 3q^{-2} + q^{-4}) \\
&\quad + a^2(q^4 - 3q^2 + 4 - 3q^{-2} + q^{-4}) \\
(q - q^{-1})P_{\mathcal{K}_0} &= a^7(q^4 + q^{-4}) - a^5(q^6 + 1 + q^{-6}) + a^3(q^2 - 1 + q^{-2}) \\
(q - q^{-1})P_{\mathcal{K}_{0-}} &= a^5(q^4 - q^2 + 2 - q^{-2} + q^{-4}) \\
&\quad + a^3(-q^6 + q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4} - q^{-6}) \\
&\quad + a(q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4})
\end{aligned}$$

Due to Rasmussen [13], there is a spectral sequence $E_k(N)$ whose E_1 term is $\overline{H}(\mathcal{K})$ and converges to $\overline{H}_N(\mathcal{K})$. When N is large we have $\overline{H}(\mathcal{K}) \simeq \overline{H}_N(\mathcal{K})$. Since $\overline{5_2}$ and \mathcal{K}_{0-} are KR-thin, we can explicitly compute $\overline{H}_N(\mathcal{K}_{0-})$, $\overline{H}_N(\overline{5_2}) \otimes M$ and $\overline{H}_N(\overline{5_2})$ from their HOMFLYPT polynomials. By the exact sequences (2.1) and (2.2), we guess possible generators of $\overline{H}(\mathcal{K}_+)$ with suitable j -grading shifts as illustrated in Figure 3. Hollow dots (\circ) represent $\overline{H}(\mathcal{K}_{0-})$. Solid dots (\bullet) represent $\overline{H}(\overline{5_2}) \otimes M$. Squares (\square) represent $\overline{H}(\overline{5_2})$.

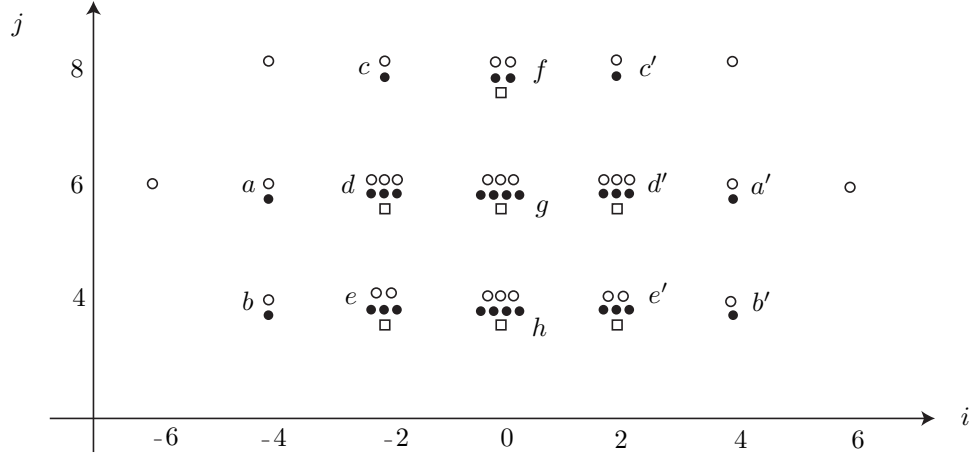


FIGURE 3. Possible generators of $\overline{H}(\mathcal{K}^*)$. The δ -gradings are $\delta(\circ) = \delta(\square) = 2$, $\delta(\bullet) = 4$.

We introduce the following claim, whose proof will be given shortly:

Claim 2.2. The two generators at b in Figure 3 survive.

Assuming Claim 2.2 and by the polynomial (2.3), we get $\delta_-(\mathcal{K}^*) = 4$ and $\delta_+(\mathcal{K}^*) = 8$. Thus the KR-MFW inequality is sharp on \mathcal{K}^* . \square

It remains to establish Claim 2.2.

Proof of Claim 2.2. According to [15], the Poincare polynomial of the reduced Khovanov homology $\overline{H}_{N=2}(\mathcal{K}^*)$ is

$$(2.4) \quad q^4 + q^4 t + q^6 t^2 + q^8 t^2 + q^8 t^3 + q^{10} t^3 + 2q^{10} t^4 + q^{12} t^5 + q^{14} t^5 + 2q^{14} t^6 + q^{16} t^7 + q^{18} t^8 + q^{20} t^9.$$

Rasmussen [13] proved that

$$\overline{H}_N^{I,J}(\mathcal{K}) \simeq \bigoplus_{\substack{i+Nj=I \\ (k-j)/2=J}} \overline{H}^{i,j,k}(\mathcal{K})$$

where I is the q -degree and J the homological degree. The only possible generators in Figure 3 corresponding to the term $q^4 + q^4 t$ of the polynomial (2.4) is the two at position b since $I = 4 = -4 + 8 = i + 2j$. \square

3. PROOF OF THEOREM 1.7

Lemma 3.1. *Suppose that $D \in \mathcal{B}_{\mathcal{K}}$ is a closed braid diagram of \mathcal{K} with $b_D = b_{\mathcal{K}}$. Focus on one site of D and construct D_+, D_-, D_0 (one of the three must be D). Let $\alpha, \beta, \gamma \in \{+, -, 0\}$ be mutually distinct. Suppose $D_\alpha = D$.*

If positive destabilization is applicable p -times to each of D_β, D_γ , then

$$(3.1) \quad (w_D + b_D - 1) - \delta_+(\mathcal{K}) \geq 2p.$$

If negative destabilization is applicable n -times to each of D_β, D_γ , then

$$(3.2) \quad \delta_-(\mathcal{K}) - (w_D - b_D + 1) \geq 2n.$$

Therefore, if $p + n > 0$ the KR-MFW inequality is not sharp on \mathcal{K} .

Proof. Let $\mathcal{K}_\alpha, \mathcal{K}_\beta, \mathcal{K}_\gamma$ be the topological knot types of $D_\alpha, D_\beta, D_\gamma$ respectively. Thanks to Rasmussen's skein exact sequence of KR-homologies (Proposition 7.6 in [13]), we have

$$\begin{aligned} \min\{\delta_-(\mathcal{K}_-) + 2, \delta_-(\mathcal{K}_0) + 1\} &\leq \delta_-(\mathcal{K}_+) \leq \delta_+(\mathcal{K}_+) \leq \max\{\delta_+(\mathcal{K}_-) + 2, \delta_+(\mathcal{K}_0) + 1\} \\ \min\{\delta_-(\mathcal{K}_+) - 2, \delta_-(\mathcal{K}_0) - 1\} &\leq \delta_-(\mathcal{K}_-) \leq \delta_+(\mathcal{K}_-) \leq \max\{\delta_+(\mathcal{K}_+) - 2, \delta_+(\mathcal{K}_0) - 1\} \\ \min\{\delta_-(\mathcal{K}_+) - 1, \delta_-(\mathcal{K}_-) + 1\} &\leq \delta_-(\mathcal{K}_0) \leq \delta_+(\mathcal{K}_0) \leq \max\{\delta_+(\mathcal{K}_+) - 1, \delta_+(\mathcal{K}_-) + 1\}. \end{aligned}$$

Suppose $D_\alpha = D_+$, $D_\beta = D_-$ and $D_\gamma = D_0$. Let \tilde{D}_- (resp. \tilde{D}_0) be a closed braid diagram obtained by applying positive destabilizations p times to D_- (resp. D_0). Then by the KR-MFW inequality we have

$$\begin{aligned} \delta_+(\mathcal{K}_-) + 2 &\leq (w_{\tilde{D}_-} + b_{\tilde{D}_-} - 1) + 2 \\ &= \{(w_{D_-} + b_{D_-} - 1) - 2p\} + 2 \\ &= (w_{D_+} - 2) + b_{D_+} - 1 - 2p + 2 \\ &= (w_{D_+} + b_{D_+} - 1) - 2p, \end{aligned}$$

and

$$\begin{aligned} \delta_+(\mathcal{K}_0) + 1 &\leq (w_{\tilde{D}_0} + b_{\tilde{D}_0} - 1) + 1 \\ &= (w_{D_0} + b_{D_0} - 1 - 2p) + 1 \\ &= (w_{D_+} - 1) + b_{D_+} - 1 - 2p + 1 \\ &= (w_{D_+} + b_{D_+} - 1) - 2p. \end{aligned}$$

Thus,

$$\delta_+(\mathcal{K}_+) \leq \max\{\delta_+(\mathcal{K}_-) + 2, \delta_+(\mathcal{K}_0) + 1\} \leq (w_{D_+} + b_{D_+} - 1) - 2p.$$

This is the inequality (3.1) of the lemma. When $D_\alpha = D_-$ or D_0 , the same argument works.

Similarly, the inequality (3.2) holds. \square

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Let $n := \sigma_n$ and $\bar{n} := \sigma_n^{-1} \in B_3$ where $n = 1, 2$. By braid isotopy and destabilizations, define closed braid diagram M_+, M_-, M_0 by;

$$\begin{aligned} M_+ (= BM_{x,y,z,w}) &:= 2^x 3^y \bar{1} \bar{2} 2^z 1^w 2 3 2 \bar{1}, \\ M_- &:= 2^x 3^y \bar{1} \bar{2} 2^z 1^w 2 3 2 1 \xrightarrow{+} 1^x 2^{y+1} 1^2 2^{z+1} 1^w 2, \\ M_0 &:= 2^x 3^y \bar{1} \bar{2} 2^z 1^w 2 3 2 \xrightarrow{+} 2^y 1^{z+1} 2^{x+1} 1^{w+1} 2. \end{aligned}$$

Here $\xrightarrow{+}$ means a positive destabilization and braid isotopy. Note that $b_{M_-} = b_{M_0} = 3$.

In the proof of Theorem 2.8 in [7], it is proved that there exist infinitely many (x, y, z, w) 's such that the braid index of the topological knot type of $BM_{x,y,z,w}$ is 4. Thus Theorem 1.7 follows from Lemma 3.1. \square

Remark 3.2. As in Figure 14 of [13], Rasmussen explicitly computes the reduced KR-homology $\overline{H}(9_{42})$ of 9_{42} and we can see that the KR-MFW inequality is not sharp on 9_{42} .

4. MAXIMAL BENNEQUIN NUMBERS OF \mathcal{K}_k AND \mathcal{L}_k

In this section, we prove Theorems 1.11 and 1.12 together. The next lemma is essentially due to Elrifai [4].

Lemma 4.1. *HOMFLYPT polynomial of \mathcal{K}_k (resp. \mathcal{L}_k) coincides with the one for the $(2, 6k+1)$ -torus knot $T_{2,6k+1}$ (resp. $T_{2,6k+5}$):*

$$P_{\mathcal{K}_k}(a, q) = P_{T_{2,6k+1}}(a, q), \quad P_{\mathcal{L}_k}(a, q) = P_{T_{2,6k+5}}(a, q).$$

Proof of Theorems 1.11 and 1.12. By Theorem 1.5 and Proposition 1.10, it follows that Conjectures 1.8 and 1.9 hold for the class B'_3 .

Knots $\mathcal{K}_k, \mathcal{L}_k$ have the following quasipositive closed braid representations;

$$\begin{aligned} D_{\mathcal{K}_k} &:= \bar{2} (1 \ 2 \ 2 \ 1 \ \bar{2})^{2k} 1, \\ D_{\mathcal{L}_k} &:= (1 \ 2 \ 2 \ 1 \ \bar{2})^{2k-1} (1 \ 2 \ 2 \ 1)^2 1. \end{aligned}$$

Since $(2 \ 1 \ \bar{2})$ and $(\bar{2} \ 1 \ 2)$ are quasi positive factors, diagram $D_{\mathcal{K}_k}$ has $6k$ quasi positive factors and $D_{\mathcal{L}_k}$ has $6k+6$.

Let $g_4(\mathcal{K})$ be the slice genus of $\mathcal{K} \subset S^3$. If \mathcal{K} has a quasipositive representative, say,

$$D = (w_1 \sigma_{j_1} w_1^{-1}) (w_2 \sigma_{j_2} w_2^{-1}) \cdots (w_p \sigma_{j_p} w_p^{-1}),$$

then thanks to Rudolph [14] we have $2 g_4(\mathcal{K}) = p - b_D + 1$. Therefore,

$$\begin{aligned} 2 g_4(\mathcal{K}_k) &= 6k - 3 + 1 = 6k - 2, \\ 2 g_4(\mathcal{L}_k) &= (6k + 6) - 3 + 1 = 6k + 4. \end{aligned}$$

Next recall Rudolph's *slice-Bennequin Inequality* [14];

$$2 g_4(\mathcal{K}) \geq w_D - b_D + 1 = \beta_D + 1 \quad \text{for } D \in \mathcal{B}_{\mathcal{K}}.$$

On each $D_{\mathcal{K}_k}, D_{\mathcal{L}_k}$, the inequality is sharp;

$$\begin{aligned} 2 g_4(\mathcal{K}_k) &= 6k - 2 = w_{D_{\mathcal{K}_k}} - b_{D_{\mathcal{K}_k}} + 1, \\ 2 g_4(\mathcal{L}_k) &= 6k + 4 = w_{D_{\mathcal{L}_k}} - b_{D_{\mathcal{L}_k}} + 1. \end{aligned}$$

Since we know that $D_{\mathcal{K}_k}, D_{\mathcal{L}_k}$ have $b_{\mathcal{K}_k} = b_{D_{\mathcal{K}_k}} = 3$ and $b_{\mathcal{L}_k} = b_{D_{\mathcal{L}_k}} = 3$ (the minimal braid index), Conjecture 1.9 holds for \mathcal{K}_k and \mathcal{L}_k .

By Lemma 4.1, we can compute the minimal and the maximal a -degrees of $P_{\mathcal{K}_k}(a, q)$ and $P_{\mathcal{L}_k}(a, q)$:

$$\begin{aligned} d_-(\mathcal{K}_k) &= 6k, & d_+(\mathcal{K}_k) &= 6k + 2, \\ d_-(\mathcal{L}_k) &= 6k + 4, & d_+(\mathcal{L}_k) &= 6k + 6. \end{aligned}$$

Since $w_{D_{\mathcal{K}_k}} + b_{D_{\mathcal{K}_k}} - 1 = 6k + 2$, one of the MFW inequalities $d_+ \leq w_D + b_D - 1$ is sharp on \mathcal{K}_k . This sharpness combined with the above argument about the slice Bennequin inequality, we conclude that Conjectures 1.8 and 1.9 hold for \mathcal{K}_k and their mirror image $\overline{\mathcal{K}_k}$.

We remark that since $w_{D_{\mathcal{L}_k}} + b_{D_{\mathcal{L}_k}} - 1 = 6k + 8 > 6k + 6 = d_+(\mathcal{L}_k)$ (the MFW inequality is not sharp on \mathcal{L}_k), the same argument does not apply to \mathcal{L}_k . \square

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